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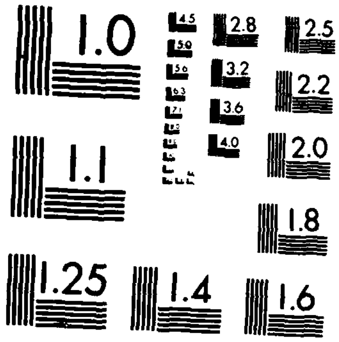

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A Technical Report

Grant No. N00014-86-K-0742

September 1, 1985 - August 31, 1986

RELATIONS BETWEEN ARRIVAL AND TIME AVERAGES OF A PROCESS IN DISCRETE-TIME SYSTEMS AND SOME APPLICATIONS

Submitted to:

Office of Naval Research
800 N. Quincy St.
Arlington, VA 22217-5000

Attention: R. N. Madan
Code 1114SE

Submitted by:

L. Georgiadis
Research Assistant Professor

P. Kazakos
Professor

Report No. UVA/525415/EE87/102
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SCHOOL OF ENGINEERING AND APPLIED SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA 22901

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We consider a process observed by arrivals in a discrete-time system. The arrivals are controlled by an underlying Markov chain. The relation between the time average of the observed process and the average of the same process as observed by the arrivals is derived. Applications of the results are provided.

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1. INTRODUCTION

The fact that "Poisson Arrivals See Time Averages" (PASTA) has been used repeatedly in the analysis of queueing systems. Various authors provided proofs of PASTA under varying assumptions on the observed process and its relationship to the Poisson arrivals [6], [7], [9]. In [10] it was shown that PASTA is essentially a sample path property. The basic condition is that the observed process cannot anticipate future jumps of the Poisson process. In this paper we considers discrete-time systems. These systems arise naturally in the study of synchronized communication networks and can also be considered as approximations of continuous-time systems. We assumed that the statistics of the arrival process are governed by an underlying denumerable Markov chain and that the observed process cannot anticipate future arrivals if the current state of the Markov chain is known. We derive the relation between the time average of the observed process and the average of the process as observed by the arrivals. The states of the underlying Markov chain are involved in this relation. Examples are given to illustrate the applicability of the results. As in [10] the results can be applied even if the observing process has a role different than "arrival."

2. MAIN RESULTS

On a probability space $(\Omega, \mathcal{F}, P)$ consider:

i) An increasing family of σ-fields, $\mathcal{F}_n$, $n=0,1,\ldots$

ii) A sequence U_n, $n=0,1,\ldots$, of random variables adapted to $\mathcal{F}_n$ (i.e. U_n is $\mathcal{F}_n$ measurable for every n).

iii) A sequence Θ_n, $n=1,2,\ldots$, of nonnegative, integer valued random variables adapted to $\mathcal{F}_n$

iv) A sequence X_n, $n=0,1,\ldots$ of random variables adapted to $\mathcal{F}_n$, such that:

$$E\{\Theta_{n+1}/\mathcal{F}_n\}=E\{\Theta_{n+1}/X_n\} \ a.e., \ n=0,1,\ldots \tag{1}$$

Let $I(A)$, $A \in \mathcal{F}$, denote the indicator function of the event A.

The interpretation of the quantities defined above is the following:

i) $\mathcal{F}_n$ is the history of the system up to time n.

ii) U_n is the observed process. U_n is usually an indicator function. The process of interest is, say, D_n, $n=0,1,\ldots$, and $U_n=I(D_n \in B)$, where B is an event in the state space of D_n.

iii) Θ_n represents bulk arrivals.

iv) X_n is the process governing the evolution of the arrival process.

Let us define:

$$T_n=\frac{\sum_{l=0}^{n} U_l}{n} \tag{2.a}$$

$$O_n^a=\frac{\sum_{l=0}^{n} U_l\Theta_l}{\sum_{l=1}^{n}\Theta_l} \tag{2.b}$$

$$O_n^b=\frac{\sum_{l=1}^{n} U_{l-1}\Theta_l}{\sum_{l=1}^{n}\Theta_l} \tag{2.c}$$

$$S_{i,n}^{a}=\frac{\sum_{l=0}^{n}U_l I(X_l=i)}{\sum_{l=0}^{n}I(X_l=i)} \tag{2.d}$$

$$S_{i,n}^{b}=\frac{\sum_{l=1}^{n}U_{l-1} I(X_l=i)}{\sum_{l=1}^{n}I(X_l=i)} \tag{2.e}$$

T_n is the time average of the observed process up to time n.

O_n^a is the average value of U_n that arrivals up to time n see, just *after* they arrive.

O_n^b is the average value of the process U_n that arrivals up to time n see, at the last slot *before* they arrive.

$S_{i,n}^a$, $S_{i,n}^b$, have the same interpretations as O_n^a, O_n^b respectively, if we consider that an arrival occurs whenever $X_n=i$, i fixed.

An example where the arrivals have the structure described above is the output of the finite population slotted ALOHA system ([4], chapter 8). In this case,

$$\Theta_n = \begin{cases} 1 & \text{if } a \textit{ successful transmission occurs in slot } n \\ 0 & \textit{otherwise} \end{cases}$$

and X_n is the number of blocked users in the beginning of slot n. The output of Tree, Window or Stack type Random Access Algorithms can be put in this framework as well. The same is true for the output of other discrete-time queues. The process D_n can be, for example, the length of a queue whose input is the output of a Random Access Algorithm. We are interested in the relationship between the quantities defined in (2), as n increases to infinity.

The following Theorem is the basis for the subsequent derivations.

Theorem 1. *Let* $|U_n|\leqslant B<\infty$, $n=1,2,\ldots$, *and* $\sum_{n=1}^{\infty}\frac{1}{n^2}E\{\Theta_n^2\}<\infty$. *Then,*

$$\lim_{n\to\infty}\left(\frac{\sum_{l=1}^{n}U_{l-1}\Theta_l}{n}-\frac{\sum_{l=1}^{n}U_{l-1}E\{\Theta_l/X_{l-1}\}}{n}\right)=0 \ a.e.$$

Proof. The proof is similar to the martingale proof of the Strong Law of Large Numbers. We include it here for completeness.

Let

$$C_n=U_{n-1}\Theta_n-U_{n-1}E\{\Theta_n/X_{n-1}\},\ n=1,2,\ldots$$

and

$$G_n=\sum_{l=1}^{n}\frac{1}{l}C_l=\begin{cases} C_1 & n=1 \\ \frac{1}{n}C_n+G_{n-1} & n=2,3,\ldots \end{cases}$$

The random variables C_n, $n=1,2,\ldots$ are integrable:

$$E\{|C_n|\}\leqslant 2B(E\{\Theta_n^2\})^{1/2}<\infty.$$

It follows that G_n is integrable for $n=1,2,\ldots$ It can also be seen that $\{G_n,\mathcal{F}_n,\ n=1,2,\ldots\}$ is a martingale. Moreover,

$$E\{G_{n+1}^2/\mathcal{F}_n\}=G_n^2+\frac{1}{(n+1)^2}E\{C_{n+1}^2/\mathcal{F}_n\}+2G_n\frac{1}{n+1}E\{C_{n+1}/\mathcal{F}_n\}$$

$$\leq G_n^2+\frac{1}{(n+1)^2}B^2E\{(|\Theta_{n+1}|+E\{|\Theta_{n+1}|/X_n\})^2/\mathcal{F}_n\}$$

$$\leq G_n^2+\frac{2B^2}{(n+1)^2}(E\{\Theta_{n+1}^2/\mathcal{F}_n\}+E\{\Theta_{n+1}^2/X_n\})$$

Therefore,

$$E\{G_n^2\}\leq E\{G_1^2\}+4B^2\sum_{l=1}^{\infty}\frac{1}{(l+1)^{2)}}E\{\Theta_{l+1}^2\}<\infty,\ n=1,2,...$$

It follows ([1], Th. 7.6.10, p301) that G_n is uniformly integrable and converges almost everywhere (and in L^2) to a finite limit $G(\infty)$. Therefore,

$$\sum_{l=1}^{\infty}\frac{1}{l}C_l=G(\infty)<\infty\ a.e.$$

The theorem now follows by an application of Kronecker's Lemma ([1], Th 7.1.3, p270). □

The following Corollary is the counterpart of the corresponding Theorem for Poisson arrivals in continous time [10].

Corollary 1. *Let* Θ_n, $n=1,2,...$ *be a sequence of random variables such that* Θ_n *is independent of* $\mathcal{F}_{n-1}$, $n=1,2,...$, $E\{\Theta_n\}=\lambda_n$, *and* $\sum_{l=1}^{\infty}\frac{1}{l^2}E\{\Theta_l^2\}<\infty$. *If*

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda_l}{n}=\lambda<\infty,$$

then

$$O_n^b\to O_\infty^b\ a.e.\ \text{ iff }\ \frac{\sum_{l=1}^{n}U_{l-1}\lambda_l}{n}\to\lambda O_\infty^b\ a.e.$$

Therefore, if $\lambda_n=\lambda$, $n=1,2,...$, *then*,

$$O_n^b\to O_\infty^b\ a.e.\ \text{ iff }\ T_n\to T_\infty=O_\infty^b\ a.e.$$

Proof. In Theorem 1., let $X_n=\Theta_n$, $n=1,2,...$, $X_0=constant$. Then

$$\lim_{n\to\infty}\left|\frac{\sum_{l=1}^{n}U_{l-1}\Theta_l}{\sum_{l=1}^{n}\Theta_l}\frac{\sum_{l=1}^{n}\Theta_l}{n}-\frac{\sum_{l=1}^{n}U_{l-1}\lambda_l}{n}\right|=0\ a.e.$$

By Kolmogorov's Strong Law of Large Numbers ([1], Th. 7.2.2, p274),

$$\frac{\sum_{l=1}^{n}\Theta_l-\sum_{l=1}^{n}\lambda_l}{n}\to 0\ a.e.$$

The rest of the Corollary follows easily. □

Remark. Note that it is not required that the random variables Θ_n be identically distributed. On the other hand if the random variables Θ_n are independent identically distributed (i.i.d), the restriction on the second moments is not necessary.

Corollary 2. *If Θ_n are i.i.d. random variables with finite mean, then*

$$O_n^b \to O_\infty^b \ a.e. \ \text{ iff } \ T_n \to T_\infty = O_\infty^b$$

Proof. The proof is entirely analogous to the corresponding proof of the Strong Law of Large Numbers ([1], Th 7.2.5, p275) and will be omitted. We only note that in the proof, instead of refering to the Strong Law of Large Numbers with finite second moments, one should refer to Theorem 1. □

We now state the conditions for the main Corollary concerning the nonindependent case.

Let X_n, $n=1,2,\ldots$, be an irreducible, homogeneous, denumerable Markov chain with state space $\mathcal{L}$, and transition probabilities $p_{ij}=P(X_{n+1}=j/X_n=i)$. Let $\mathcal{H}$ be the state space of Θ_n, $\mathcal{H}\subseteq\{0,1,2,\ldots\}$. Let $D_i=E\{\Theta_{n+1}/X_n=i\}$, independent of n, and $M_i=E\{\Theta_{n+1}^2/X_n=i\}$, independent of n.

Corollary 3. *Let* $|U_n|\leq B<\infty$ *a.e., $n=1,2,\ldots$ Let X_n be ergodic with stationary probabilities* π_i, $i\in\mathcal{L}$. *If*

$$M_i \leq M < \infty, \ i \in \mathcal{L} \tag{3}$$

and

$$S_{i,n}^a \to S_{i,\infty}^a \ a.e., \ i \in \mathcal{L}$$

then

$$S_{i,n}^b \to S_{i,\infty}^b \ a.e., \ i\in\mathcal{L}, \ O_n^b \to O_\infty^b \ a.e., \ T_n \to T_\infty \ a.e.$$

and the following equalities hold:

$$O_\infty^b\Big(\sum_{i\in\mathcal{L}} D_i\pi_i\Big) = \sum_{i\in\mathcal{L}} D_i\pi_i S_{i,\infty}^a, \ a.e. \tag{4.a}$$

$$S_{j,\infty}^b \pi_j = \sum_{i\in\mathcal{L}} \pi_i p_{ij} S_{i,\infty}^a, \ \ i,j\in\mathcal{L}, \ a.e. \tag{4.b}$$

$$T_\infty = \sum_{i\in\mathcal{L}} \pi_i S_{i,\infty}^b = \sum_{i\in\mathcal{L}} \pi_i S_{i,\infty}^a, \ a.e. \tag{4.c}$$

Proof. From (3) it follows that

$$E\{\Theta_n^2\} \leq M, \ n=1,2,\ldots \tag{5}$$

Therefore, Theorem 1 applies with $U_n\equiv 1$.

$$\lim_{n\to\infty}\left|\frac{\sum_{l=1}^{n}\Theta_l}{n} - \frac{\sum_{l=1}^{n} E\{\Theta_l/X_{l-1}\}}{n}\right| = 0 \tag{6}$$

Because of (3), $\sup_{i\in\mathcal{L}} D_i<\infty$, and therefore,

$$\sum_{i\in\mathcal{L}} D_i\pi_i < \infty \tag{7}$$

From (7) and Th.2 in [3] p92, it follows that

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n} E\{\Theta_l/X_{l-1}\}}{n} = \sum_{i\in\mathcal{L}} D_i\pi_i, \ a.e. \tag{8}$$

From (6) and (8) we see that

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\Theta_l}{n}=\sum_{i\in\mathcal{L}}D_i\pi_i, \; a.e. \tag{9}$$

Observe now that $E\{\Theta_{n+1}/X_n\}=\sum_{i\in\mathcal{L}}D_i I(X_n=i)$, and apply Theorem 1 again:

$$\lim_{n\to\infty}\left|\frac{\sum_{l=1}^{n}U_{l-1}\Theta_l}{\sum_{l=1}^{n}\Theta_l}\,\frac{\sum_{l=1}^{n}\Theta_l}{n}-\frac{\sum_{i\in\mathcal{L}}\sum_{l=1}^{n}U_{l-1}D_i I(X_{l-1}=i)}{n}\right|=0 \; a.e. \tag{10}$$

Also,

$$\frac{\sum_{l=1}^{n}U_{l-1}I(X_{l-1}=i)}{n}=\frac{\sum_{l=1}^{n}U_{l-1}I(X_{l-1}=i)}{\sum_{l=1}^{n}I(X_{l-1}=i)}\,\frac{\sum_{l=1}^{n}I(X_{l-1}=i)}{n}\xrightarrow[n\to\infty]{}S^a_{i,\infty}\pi_i \; a.e. \tag{11}$$

Let A_m be an arbitrary finite subset of $\mathcal{L}$. Since

$$|U_{l-1}D_i I(X_{l-1}=i)|\leq BD_i I(X_{l-1}=i), \tag{12}$$

it can be seen that,

$$\left|\frac{\sum_{i\in\mathcal{L}}\sum_{l=1}^{n}U_{l-1}D_i I(X_{l-1}=i)}{n}-\frac{\sum_{i\in A_m}D_i\sum_{l=1}^{n}U_{l-1}I(X_{l-1}=i)}{n}\right|\leq B\frac{\sum_{l=1}^{n}\sum_{i\in A_m^c}D_i I(X_{l-1}=i)}{n} \tag{13}$$

Exactly as in the proof of (8), we have that,

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\sum_{i\in A_m^c}D_i I(X_{l-1}=i)}{n}=\sum_{i\in A_m^c}D_i\pi_i \; a.e. \tag{14}$$

From (11), (13), (14), (7) and the arbitrariness of A_m we conclude that,

$$\lim_{n\to\infty}\frac{\sum_{i\in\mathcal{L}}\sum_{l=1}^{n}U_{l-1}I(X_{l-1}=i)}{n}=\sum_{i\in\mathcal{L}}D_i\pi_i S^a_{i,\infty} \; a.e. \tag{15}$$

Formula (4.a) is established by combining (9), (10) and (15). Formula (4.b) follows immediately by setting $\Theta_n=I(X_n=j)$. To prove formula (4.c) observe that

$$U_n=\sum_{i\in\mathcal{L}}U_n I(X_{n+1}=i)=\sum_{i\in\mathcal{L}}U_n I(X_n=i) \; a.e.$$

and apply the arguments used to prove (15). □

Remarks. i) If D_i is constant, it follows from (4.a) and (4.c) that $T_\infty=O^b_\infty$. Therefore, in this case arrivals see time averages. Note that independence is not necessary.
ii) A particularly simple relation holds if $\mathcal{L}=\{0,1\}$ and $\Theta_n=I(X_n=1)$. In this case we have from (4.b),

$$O^b_\infty\pi^1=S^b_{1,\infty}\pi_1=\pi_1 p_{11}S^a_{1,\infty}+\pi_0 p_{01}S^a_{0,\infty}$$

$$=\pi_1 p_{11}O^a_{1,\infty}+\pi_0 p_{01}S^a_{0,\infty} \; a.e.$$

Also, from (4.c),

$$T_\infty=\pi_1 O^a_{1,\infty}+\pi_0 S^a_{0,\infty} \ a.e.$$

Therefore,

$$O^b_\infty-(p_{11}-p_{01})O^a_\infty=\frac{p_{01}}{\pi_1}T_\infty \ a.e. \tag{16}$$

Formula (16) involves only the quantities seen by arrivals and the time average of the process. It will be used in Section 3.2.

iii) The quantities O^a_n, O^b_n, represent the averages of the process observed by bulk arrivals. In practice the average of the process observed by a particular user is of interest. Specifically, for l arrivals, the quantities of interest are:

$$O^a_l=\frac{\sum_{m=1}^{l} U_{T_m}}{l} \quad , \quad O^b_l=\frac{\sum_{m=1}^{l} U_{T_m-1}}{l}$$

where T_m is the time of the lth arrival.

Under the conditions of Corollary 3, it can be shown using standard ratio limit arguments, that,

$$\lim_{n\to\infty} O^a_n=O^a_\infty \ a.e. \quad \textit{iff} \quad \lim_{l\to\infty} O^a_l=O^a_\infty=O^a_\infty \ a.e.$$

and

$$\lim_{n\to\infty} O^b_n=O^b_\infty \ a.e. \quad \textit{iff} \quad \lim_{l\to\infty} O^b_l=O^b_\infty=O^b_\infty \ a.e.$$

If the Θ_n are i.i.d., the restriction on the second moments is not necessary.

iv) The formulation in this section was in terms of sample averages. Results can also be obtained in terms of limiting probabilities. Consider the following example, useful in applications. Let $T_m, m=1,2,...$ be a sequence of random variables taking positive, integer values. Let $T_m, m=1,2,...$ be independent of $D_n, n=1,2,...$. If $\lim_{m\to\infty} T_m=\infty \ a.e.$ and

$$\lim_{n\to\infty} P(D_n=l)=P_l,$$

then

$$\lim_{m\to\infty} P(D_{T_m}=l)=P_l$$

The result follows by observing that

$$P(D_{T_m}=l)=\sum_{n=1}^{\infty} P(D_n=l \bigcap T_m=n)=\sum_{n=1}^{\infty} P(D_n=l)P(T_m=n)$$

and

$$\lim_{m\to\infty} P(T_m=n)=0, \ n=1,2,...$$

Let now $D_n, n=1,2,...$ be independent of $\Theta_n, n=1,2,...$ and let T_m be the time of the mth arrival. Then D_{T_m} is the value of D_n observed by the mth arrival. If $\lim_{m\to\infty} T_m=\infty \ a.e.$, which is usually the case in practical systems, then the limiting probabilities of D_{T_m} and D_n are the same. Note that no assumptions on the statistics of the two processes are required.

3. APPLICATIONS

3.1 The discrete-time GI|G|c queue.

Consider a communication node consisting of c servers. Time is divided in intervals of constant length, called slots. The unit of time is the length of a slot. Slot $n, n=1,2,...$,

occupies the time interval $[n-1, n)$. During a time slot n, a number of messages Θ_n arrives in the node. The length of message i is Z_i slots. The processes Θ_n and Z_i consist of i.i.d. random variables, and are independent.

Define:

N_n: The size of the queue at the beginning of slot n (just after the arrivals Θ_n).

I_n: The number of messages whose service is completed in slot n.

N_m^b: The value of N_n in the slot just prior to the mth arrival.

M_m^b: The value of $M_n = N_n - I_n$ in the slot just prior to the mth arrival.

Let

$$\mathcal{F}_n = \mathcal{F}(\Theta_l, 0 \leqslant l \leqslant n, \; Z_l, 1 \leqslant l < \infty, \; N_0) \tag{17}$$

If $E\{\Theta_n\} < cE\{Z_i\}^{-1}$, ergodic conditions hold. Considering the system in steady state, we have from Corollary 2

$$P(N_m^b = l) = P(N_n = l), \; l = 0,1,... \tag{18}$$

and

$$P(M_m^b = l) = P(M_n = l), \; l = 0,1,... \tag{19}$$

Since $M_n \geqslant N_n$, we have that

$$P(M_n \leqslant l) \leqslant P(N_n \leqslant l), \; l = 0,1,... \tag{20}$$

Also, $N_{n+1} = M_n + \Theta_n \geqslant M_n$. Therefore,

$$P(N_{n+1} \leqslant l) = P(N_n \leqslant l) \leqslant P(M_n \leqslant l), \; l = 0,1,... \tag{21}$$

From (20) and (21) we conclude that

$$P(M_n = l) = P(N_n = l), \; l = 0,1,... \tag{22}$$

Therefore,

$$P(M_m^b = l) = P(N_n = l), \; l = 0,1... \tag{23}$$

Property (23) was used in [2] for the analysis of the GI|G|1 discrete-time queue. It seems that in [2], M_n was confused with N_n, but as formula (22) shows, this does not alter the final result.

3.2 Star network with Markovian inputs.

This system was studied in [8]. An infinite buffer in a node accepts messages from K links. Time is slotted. In slot n, link i generates one or zero messages. The length of a message is constant, equal to one slot. One message is processed per slot by the node.

Let

$$\Theta_n^i = \begin{cases} 1 & \textit{if one message is generated at link } i \textit{ in slot } n \\ 0 & \textit{otherwise} \end{cases}$$

The sequences $\{\Theta_n^i, n = 1,2,...\}$ are homogeneous Markov chains for $1 \leqslant i \leqslant K$, and independent. Let

$$p_{rs}^i = P(\Theta_{n+1}^i = s / \Theta_n^i = r) > 0$$

$$\gamma_i = p_{11}^i - p_{01}^i$$

$$\alpha_i = \frac{p_{01}^i}{1-\gamma_i} = \pi_1^i$$

We define

N_n: The size of the queue at the beginning of slot n (just after the arrivals).

$N_{i,m}^b$:The value of N_n in the slot just prior to the mth arrival at link i.

$N_{i,m}^a$:The value of N_n in the slot of the mth arrival at link i.

$\Theta_{i,m}^j$:The number of arrivals at link j, in the slot of the mth arrival at link $i, i \neq j$.

$I_{i,m}$: The number of messages completing service in the slot of the mth arrival at link i.

$W_{i,m}$:

The delay of the mth arrival at link i.

N_n^i: The number of messages from link i in the queue, at slot n.

We will provide the formula for the mean queue lenght and, consequently, the average waiting time of a message. This formula was derived in [8]. The procedure followed in [8] relies on the computation of the characteristic function of the queue length. We present here a simplified proof that is based on probabilistic arguments. In addition, the average waiting time of a message at a particular link, and the probability of zero queue lenght in the slot just prior to the arrival of this message, are derived during the proof.

The queue discipline is first-come first-served. Let us assume for simplicity that if links $i_1, i_2, i_1 < i_2$ generate messages at the same slot, link i_1 is served first.

If $\sum_{i=1}^{K} \alpha_i < 1$, ergodic conditions hold. We consider the system in steady state. Let

$$\mathcal{F}_n = \mathcal{F}(\Theta_l^i, 0 \leqslant l \leqslant n, 1 \leqslant i \leqslant K, N_0)$$

From formula (16) we have that

$$P(N_{i,m}^b = l) - \gamma_i P(N_{i,m}^a = l) = (1-\gamma_i) P(N_n = l), \; l = 0,1,\ldots, \; 1 \leqslant i \leqslant K \tag{24}$$

Therefore

$$E\{N_{i,m}^b\} - \gamma_i E\{N_{i,m}^a\} = (1-\gamma_i) E\{N_n\}, \; 1 \leqslant i \leqslant K \tag{25}$$

Observe now, that

$$N_{i,m}^a = N_{i,m}^b + \sum_{\substack{1 \leqslant j \leqslant L \\ i \neq j}} \Theta_{i,m}^j + 1 - I_{i,m}, \; 1 \leqslant i \leqslant K \tag{26}$$

From Remark iv) in Section 2, we easily conclude that

$$P(\Theta_{i,m}^j) = \alpha_j = E\{\Theta_{i,m}^j\} \tag{27}$$

Therefore, from (26) we have that

$$E\{N_{i,m}^a\} = E\{N_{i,m}^b\} + \sum_{j=1}^{K} \alpha_j - \alpha_i + P(N_{i,m}^b = 0) \tag{28}$$

From (24) we see that

$$P(N_{i,m}^b = 0) - \gamma_i P(N_{i,m}^a = 0) = (1-\gamma_i) P(N_n = 0) \tag{29}$$

But $N_{i,m}^a \geqslant 1$ and therefore, $P(N_{i,m}^a = 0) = 0$ Also, $P(N_n = 0) = 1 - \sum_{i=1}^{K} \alpha_i$. This can be proved as in example 11-8, p400, in [5]. Although the GI|G|c queue is treated in [5], the proof goes through in our case. Therefore,

$$P(N_{i,m}^b = 0) = (1-\gamma_i) P(N_i = 0) = (1-\gamma_i)(1 - \sum_{j=1}^{K} \alpha_j) \tag{30}$$

Combining (28) and (30) we get

$$E\{N^a_{i,m}\}=E\{N^b_{i,m}\}+\sum_{j=1}^{K}\alpha_j-\alpha_i+(1-\gamma_i)(1-\sum_{j=1}^{K}\alpha_j) \tag{31}$$

Next, observe that[1]

$$W_{i,m}=N^b_{i,m}+\sum_{j=1}^{i-1}\Theta^j_{i,m}+1-I_{i,m} \tag{32}$$

Therefore,

$$E\{W_{i,m}\}=E\{N^b_{i,m}\}+\sum_{j=1}^{i-1}\alpha_j+(1-\gamma_i)(1-\sum_{j=1}^{K}\alpha_j) \tag{33}$$

By Little's formula applied to each link separately, we have that

$$\alpha_i E\{W_{i,m}\}=E\{N^i_n\},\ 1\leq i\leq K \tag{34}$$

Therefore

$$\sum_{i=1}^{K}\alpha_i E\{W_{i,m}\}=\sum_{i=1}^{K}E\{N^i_n\}=E\{N_n\} \tag{35}$$

From (25), (31) and (33) we find that

$$E\{W_{i,m}\}=E\{N_n\}+\frac{\gamma_i}{1-\gamma_i}(\sum_{\substack{j=1\\ i\neq j}}\alpha_j)+1-\sum_{j=i}^{K}\alpha_j \tag{36}$$

Finally, from (35) and (36) we conclude after some simple algebra, that

$$E\{N_n\}=\frac{\sum_{i=1}^{K}\sum_{j>i}^{K}\alpha_i\alpha_j\left[1+\frac{\gamma_i}{1-\gamma_i}+\frac{\gamma_j}{1-\gamma_j}\right]}{1-\sum_{i=1}^{K}\alpha_i}+\sum_{i=1}^{K}\alpha_i \tag{37}$$

Formula (37) is derived in [8]. Formula (36) provides the expected delay of a message arriving at link i. Formula (30) provides the probability that a message arriving at link i will find the queue at the previous slot empty.

3.3 Queues with Input Controlled by a Markov Chain.

Consider a discrete-time (slotted) queue. Let $\Theta_n, n=0,1,....$, be the number of messages generated in a slot. Let $X_n, n=0,1,....$, be a process with denumerable state space $\mathcal{L}$, such that for any nonnegative integers $n,h,r,r_1,\ldots,r_n$, and any $j,i,i_1,\ldots,i_n$ from the state space $\mathcal{L}$, the following equality holds:

$$P(\Theta_{n+1}=h,\ X_{n+1}=j\ /\ X_n=i,\ X_{n-1}=i_1,\ldots,\ X_0=i_n,\ \Theta_n=r,\ \Theta_{n-1}=r_1,\ldots,\ \Theta_0=r_n) \tag{38}$$

$$=P(\Theta_{n+1}=h,\ X_{n+1}=j\ /\ X_n=i)=q_{i(j,h)}$$

The length of message i is Z_i slots. The sequence $Z_i, i=1,2,....$, consists of i.i.d. random variables and is independent of the sequences Θ_n and X_n. The output of many Random Access Algorithms as well as the output of other discrete-time queues, has the statistical structure of Θ_n.

---

[1]We adopt the notation, $\sum_{i=a}^{b}=0$ if $a>b$.

A complete study of queues of the type described in the previous paragraph is outside the scope of the present paper. We will illustrate here how the steady state distribution of the delay is related to the steady state distribution of the queue length in the case where Z_i are geometrically distributed with parameter ρ.

$$P(Z_i=k)=\rho(1-\rho)^{k-1},\ k=1,2,... \tag{39}$$

We will also assume that Θ_n is either zero or one. The process X_n is a Markov chain with transition probabilities

$$P(X_{n+1}=j/X_n=i)=p_{ij}=q_{i(j,1)}+q_{i(j,0)} \tag{40}$$

We assume that X_n is irreducible, aperiodic and ergodic. The process (X_n, Θ_n), is also a Markov chain with transition probabilities

$$P(X_{n+1}=j,\ \Theta_{n+1}=h\ /\ X_n=i,\ \Theta_n=r)=p_{(i,r)(j,h)}=q_{i,(j,h)} \tag{41}$$

We assume that (X_n, Θ_n) is irreducible and aperiodic. Since X_n is ergodic, it follows that (X_n, Θ_n) is ergodic with stationary transition probabilities

$$\pi_{(j,h)}=\sum_{i\in\mathcal{L}} q_{i(j,h)}\pi_i,\ j\in\mathcal{L},\ h=1,0 \tag{42}$$

Formula (42) follows easily by noting that the numbers $\pi_{(j,h)}$ satisfy the equilibrium equations for the Markov chain (X_n, Θ_n), and that $\sum_{j\in\mathcal{L}}\sum_{h=1}^{2}\pi(j,h)=\sum_{j\in\mathcal{L}}\pi_j=1$.

Let N_n, $N_m^b=N_{1,m}^b$, $N_m^a=N_{1,m}^a$ be as defined in Section 3.2. Let also,

$\mathcal{N}_{(i,r),m}^a$:
The number of messages in the queue, in the slot of the mth occurrence of state (i,r) (i.e., the mth time that $X_n=i,\ \Theta_n=r$ for some n.)

$\mathcal{N}_{(i,r),m}^b$:
The number of messages in the queue, in the slot before the mth occurrence of state (i,r).

$\Theta_{(i,r),m}^a$:
The number of arrivals in the slot of the mth occurrence of state (i,r).

$J_{(i,r),m}$:
The number of messages completing service in the slot of the mth occurrence of state (i,r).

W_m: The delay of the mth arrival in the system.

To simplify the notation we will omit the indices m,n whenever there is no danger for confusion. We consider that (X_n, Θ_n) is the underlying Markov chain. Then,

$$D_{(i,r)}==E\{\Theta_{n+1}/X_n=i,\ \Theta_n=r\}=\sum_{j\in\mathcal{L}} q_{i(j,1)} \tag{43}$$

We assume that

$$\sum_{i\in\mathcal{L}}\sum_{r=1}^{2} D_{(i,r)}\pi_{(i,r)}=\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}} q_{i(j,1)}\pi_i<\rho \tag{44}$$

Under (44), the system can be considered in steady state.[2] From Corollary 3, we obtain

$$P(N^b=l)\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}} q_{i,(j,1)}\pi_i=\sum_{i\in\mathcal{L}}\sum_{r=1}^{2} D_{(i,r)}\pi_{(i,r)}P(\mathcal{N}_{(i,r)}^a=l) \tag{45.a}$$

[2] The existence of steady state can be established, for example, by the methods of Ch.1 in: Borovkov, A. A. *Stochastic Processes in Queueing Theory*. New York: Springer-Verlag 1976.

$$P(\mathcal{N}^b_{(j,h)}=l)(\sum_{i\in\mathcal{L}} q_{i(j,h)}\pi_i)=\sum_{i\in\mathcal{L}}\sum_{r=1}^{2}\pi_{(i,r)}q_{i(j,h)}P(\mathcal{N}^a_{(i,r)}=l),\ j\in\mathcal{L},\ h=0,1 \tag{45.b}$$

$$P(N=l)=\sum_{i\in\mathcal{L}}\sum_{r=1}^{2}\pi_{(i,r)}P(\mathcal{N}^a_{(i,r)}=l) \tag{45.c}$$

Observe that

$$\mathcal{N}^a_{(i,r)}=\mathcal{N}^b_{(i,r)}-J_{(i,r)}+r,\ i\in\mathcal{L},\ r=0,1 \tag{46}$$

Therefore,

$$P(\mathcal{N}^a_{(i,1)}=l) = P(\mathcal{N}^b_{(i,1)}=l-1)\,(1-\rho) + P(\mathcal{N}^b_{(i,1)}=l)\,\rho,\ l=1,2...,\ i\in\mathcal{L} \tag{47.a}$$

$$P(\mathcal{N}^a_{(i,1)}=0) = 0,\ i\in\mathcal{L} \tag{47.b}$$

$$P(\mathcal{N}^a_{(i,0)}=l) = P(\mathcal{N}^b_{(i,0)}=l)\,(1-\rho) + P(\mathcal{N}^b_{(i,0)}=l+1)\,\rho,\ l=1,2....,\ i\in\mathcal{L} \tag{47.c}$$

$$P(\mathcal{N}^a_{(i,0)}=0) = P(\mathcal{N}^b_{(i,0)}=0) + P(\mathcal{N}^b_{(i,0)}=1)\,\rho,\ i\in\mathcal{L} \tag{47.d}$$

Equations (45) and (47) relate the steady state distribution of N^b_m to the steady state distribution of N_n. Since

$$W_m = \begin{cases} Z_m & \text{if } N^b_m=0 \\ \sum_{i=1}^{N^b_m} Z_i + Z_m - 1 & \text{if } N^b_m=1,2... \end{cases} \tag{48}$$

where the random variables Z_i are independent geometrically distributed, the steady state distribution of W_m can be determined from the distribution of N^b_m

4. CONCLUSIONS

We presented the relation between the time average of a process and the average of the same process as observed by arrivals, in discrete time. The observing process is controlled by an underlying Markov chain and is usually identified with the arrival process, but the derived relations are independent of this identification. The results were applied to the study of certain discrete-time systems. It is believed that the results will facilitate the analysis of other discrete-time systems, and that they have counterparts in continuous-time systems.

REFERENCES

[1] B. R. Ash, *Real Analysis and probability*, Probability and Mathematical Statistics, Vol. II, Academic Press, New york, 1972.

[2] H. Bruneel, "Comments on Discrete-Time Queueing Systems and Their Networks," *IEEE Trans. on Communications*, vol. COM-28, No. 3, pp. 461-463, 1980.

[3] K. L. Chung, *Markov Chains with Stationary Transition Probabilities*, Springer Verlag, New York, 1967.

[4] F. J. Hayes, *Modeling and Analysis of Computer Communications Networks*, Plenum Press, New York, 1984.

[5] D. P. Heyman and M. J. Sobel, *Stochastic Models in Operations Research*, McGraw Hill, New York, 1982.

[6] S. Jr. Stidham, "Regenerative Processes in the Theory of Queues, with Applications to the Alternating Priority Queue," *Adv. Appl. Prob.*, vol. 4, pp. 542-577, 1972.

[7] R. E. Strauch, "when a Queue Looks the Same to an Arriving Customer as to an Observer," *Management Science*, vol. 17, pp. 140-141, 1970.

[8] A. M. Viterbi, "Approximate Analysis of Time-Synchronous Packet Networks," *IEEE Journal on Selected Areas in Communication*, vol. SAC-28, No. 6, pp. 879-890, 1986.

[9] R. W. Wolff, "Work-Conserving Priorities," *J. Appl. Prob.*, vol. 7, pp. 327-337, 1970.

[10] R. W. Wolff, "Poisson Arrivals See Time Averages," *Oper. Res.*, vol. 30-2, pp 223-231, 1982.

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